Cosets of affine vertex algebras inside larger structures

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Joint work with T. Creutzig (University of Alberta),

Based on arXiv:1407.8512.

Let \mathcal{V} be a vertex algebra.

 $G \subset Aut(V)$ a finite-dimensional, reductive group. Define *orbifold*

$$\mathcal{V}^G = \{ v \in \mathcal{V} | gv = v, \quad \forall g \in G \}.$$

 $\mathcal{A} \subset \mathcal{V}$ a vertex subalgebra. Define *coset*

$$\mathsf{Com}(\mathcal{A}, \mathcal{V}) = \{ v \in \mathcal{V} | [a(z), v(w)] = 0, \quad \forall a \in \mathcal{A} \}.$$

Suppose V has a nice property, such as strong finite generation, C_2 -cofiniteness, or rationality.

Problem: Do \mathcal{V}^G and $\mathsf{Com}(\mathcal{A},\mathcal{V})$ inherit this property?

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V a finite-dimensional G-module (over $\mathbb C$).

 $\mathbb{C}[V]$ ring of polynomial functions on V.

 $\mathbb{C}[V]^G$ ring of G-invariant polynomials.

Fundamental problem: Find generators and relations for $\mathbb{C}[V]^G$.

Thm: (Hilbert, 1893) $\mathbb{C}[V]^G$ is finitely generated for any G and V.

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Let V be a G-module. For $j \geq 0$, let $V_j \cong V$. Let

$$R=\mathbb{C}[\oplus_{j\geq 0}V_j]^G.$$

First fundamental theorem (FFT) for (G, V) is a set of generators for R.

Second fundamental theorem (SFT) for (G, V) is a set of generators for the ideal of relations in R.

- ► Standard representations of classical groups (Weyl, 1939)
- ▶ Adjoint representations of classical groups (Procesi, 1976),
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Generator $\theta \in \mathbb{Z}/2\mathbb{Z}$ acts on V by -1.

 x_j a basis for V_j^* for $j \ge 0$.

$$\theta(x_j) = -x_j.$$

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Heisenberg vertex algebra ${\mathcal H}$ has generator b(z) satisfying $b(z)b(w)\sim (z-w)^{-2}.$

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$$\{: \partial^{k_1}b \cdots \partial^{k_r}b : | 0 \le k_1 \le \cdots \le k_r\}.$$

$$\operatorname{Aut}(\mathcal{H})\cong \mathbb{Z}/2\mathbb{Z}$$
, generator $\theta:\mathcal{H}\to\mathcal{H}$ acts by $\theta(b)=-b$.

 $\mathcal H$ is linearly isomorphic to $\mathbb C[x_0,x_1,x_2,\dots]$ where $x_j\leftrightarrow\partial^j b$.

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$$\partial(x_j) = x_{j+1}$$
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Relations are $q_{i,j}q_{k,l} - q_{i,k}q_{j,l}$.

$$R\cong \mathcal{H}^{\mathbb{Z}/2\mathbb{Z}}$$
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$$\omega_{i,j} = : \partial^i b \partial^j b :, \qquad 0 \le i \le j.$$

Recall
$$\partial(a_{i,j}) = a_{i+1,j} + a_{i,j+1,j}$$

 $\{a_0|_{2k}|_{k}>0\}$ minimal generating set for R as a differential algebra.

 $\{\omega_{0.2k}|k\geq 0\}$ strongly generates $\mathcal{H}^{\mathbb{Z}/2\mathbb{Z}}$. But this is not minimal!

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For all $k \geq 2$, we have a decoupling relation $\omega_{0,2k} = P(\omega_{0,0}, \omega_{0,2})$.

$$\begin{aligned} \mathbf{E}\mathbf{x}: & \ \omega_{0,4} = -\frac{2}{5}: \omega_{0,0}\partial^2\omega_{0,0}: \ +\frac{4}{5}: \omega_{0,0}\omega_{0,2}: \ +\frac{1}{5}: \partial\omega_{0,0}\partial\omega_{0,0}: \\ & \ +\frac{7}{5}\partial^2\omega_{0,2} -\frac{7}{30}\partial^4\omega_{0,0}. \end{aligned}$$

Alternatively, this can be written in the form

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This is a *quantum correction* of the analogous classical relation $q_{0,0}q_{1,1}-q_{0,1}q_{0,1}$.

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$$\omega_{0,4} = -\frac{4}{5}(:\omega_{0,0}\omega_{1,1}: - :\omega_{0,1}\omega_{0,1}:) + \frac{7}{5}\partial^2\omega_{0,2} - \frac{7}{30}\partial^4\omega_{0,0}.$$

This is a *quantum correction* of the analogous classical relation $q_{0.0}q_{1.1} - q_{0.1}q_{0.1}$.



Heisenberg algebra $\mathcal{H}(n)$: even generators $b^i, i=1,\ldots,n$, $b^i(z)b^j(w)\sim \delta_{i,j}(z-w)^{-2}$.

Free fermion algebra $\mathcal{F}(n)$: odd generators $\phi^i, i=1,\ldots,n,$ $\phi^i(z)\phi^j(w)\sim \delta_{i,j}(z-w)^{-1}.$

$$eta\gamma$$
-system $\mathcal{S}(n)$: even generators $eta^i, \gamma^i, i=1,\ldots,n,$ $eta^i(z)\gamma^j(w)\sim \delta_{i,j}(z-w)^{-1}.$

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 $\mathcal{H}(n)$ and $\mathcal{F}(n)$ have full automorphism group O(n).

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Thm: (L, 2012) $S(n)^{Sp(2n)}$ is of type $W(2, 4, ..., 2n^2 + 4n)$.

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Sketch of proof: For any reductive $G \subset Aut(\mathcal{V})$, \mathcal{V}^G is a module over $\mathcal{V}^{Aut(\mathcal{V})}$.

By a theorem of Dong-Li-Mason (1996), ${\mathcal V}$ has a decompositior

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 \mathcal{V}^G is also a direct sum of irreducible $\mathcal{V}^{\mathsf{Aut}(\mathcal{V})}$ -modules.

 \mathcal{V}^G has a generating set that lies in the direct sum of *finitely many* of these modules.

SFG property of \mathcal{V}^G follows from these observations.

Let $\mathcal{V} = \mathcal{H}(n) \otimes \mathcal{F}(m) \otimes \mathcal{S}(r) \otimes \mathcal{A}(s)$ be a general free field algebra.

Let $G \subset \operatorname{Aut}(\mathcal{V})$ be any reductive group preserving the tensor factors, i.e, $G \subset O(n) \times O(m) \times Sp(2r) \times Sp(2s)$.

Cor: V^G is SFG

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$K \subset \mathbb{C}$ a subset which is at most countable.

 F_K the \mathbb{C} -algebra of rational functions

$$\frac{p(\kappa)}{q(\kappa)}$$
, $\deg(p) \le \deg(q)$,

such that the roots of q lie in K.

A deformable family \mathcal{B} is a vertex algebra defined over F_K .

For $k \notin K$, ordinary vertex algebra $\mathcal{B}_k = \mathcal{B}/(\kappa - k)$.

 $\mathcal{B}_{\infty} = \lim_{\kappa \to \infty} \mathcal{B}$ is a well-defined vertex algebra over \mathbb{C} .

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13. Examples

Let $\mathfrak g$ be a reductive Lie algebra with a nondegenerate form \langle,\rangle .

 $V^k(\mathfrak{g})$ the corresponding universal affine vertex algebra.

Exists deformable family $\mathcal V$ with $K=\{0\}$ satisfying

$$V_k = V/(\kappa^2 - k) \cong V^k(\mathfrak{g}), \qquad k \neq 0$$

$$\mathcal{V}_{\infty} = \lim_{\kappa \to \infty} \mathcal{V} \cong \mathcal{H}(n)$$
, where $n = \dim(\mathfrak{g})$

Let $G \subset \operatorname{Aut}(V^k(\mathfrak{g}))$ be a reductive group.

We have $\lim_{\kappa\to\infty} \mathcal{V}^G \cong \mathcal{H}(n)^G$.

Cor: $V^k(\mathfrak{g})^G$ is SFG for generic values of k



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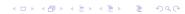
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Exists deformable family ${\mathcal V}$ with ${\mathcal K}=\{0\}$ satisfying

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Thm: (Creutzig-L, 2014) For all $\mathfrak{g}, \mathfrak{g}', \mathcal{C}^k = \text{Com}(V^k(\mathfrak{g}), V^k(\mathfrak{g}'))$ is SFG for generic values of k.

Idea of proof: $\lim_{k\to\infty} \mathcal{C}^k \cong \mathcal{H}(m-n)^G$. Here G is a reductive group with Lie algebra \mathfrak{g} .

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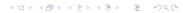
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